MATHEMATICS

ON THE TEACHING OF SECONDARY MATHEMATICS

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By "secondary mathematics" I understand geometry, plane and solid, trigonometry, algebra, including the theory of quadratics, logarithms, progressions, the elements of permutations and combinations, the binomial theorem, the principles of indetermined coefficients, and a small amount of the theory of equations. This is more than the high school has undertaken in the past; but it has become necessary to include so much through the action of many colleges and scientific schools that require as much, or allow these advanced subjects to be presented as entrance subjects.

As I see it, the principal object for us secondary-school teachers to aim at is to interest the pupil and to awaken his love for the subject. We cannot flatter ourselves with the hope of being able to do this with every pupil; but with a large number it can be done, and the results, where this is accomplished, are so fine that I can conceive of no worthier object of a teacher's ambition. I owe, in my own experience, an abiding love for mathematics—a love which has been an impelling force—to an unusually stimulating teacher that I was fortunate enough to have at a decisive period of my life; a man that took me, so to say, on a high mountain and showed me a land of promise.

The idea that a love for mathematics can be awakened in many boys would seem quixotic to the public at large. The impression is widely spread that the mathematical faculty is a peculiar gift which is found in but few of the elect. That the results attained in the past in many colleges and schools have given color to this idea is unfortunately true.

The reasons are several. First: too low a standard of scholarship and pedagogical preparation has prevailed in the past.

A teacher that has not gone much farther than his pupils in the studies that he has to teach them cannot be anything but a poor mechanical drudge. Yet there must be many such even now; else the publishing companies would not find the issuing of complete keys to their elementary mathematical textbooks—algebra ponies, if you please—a paying business. The second reason, and a consequence of the first, is the so-called recitation method, the assigning of a lesson from the book, so many pages ahead, and the next day the hearing of the lesson from the book. There is no remedy for this particular evil but insistence on thorough academic preparation; for to the teacher who is master of his subject these monstrosities are impossible. But even the teacher who knows his business is much handicapped by the conventional sequence of subjects—the air-tight compartment system, as it has not inaptly been called. It is hard to break away, and there are severe penalties attached to such eccentricity.

It is usual to spend the first year of the high-school course in algebra, then another year in plane geometry, the third year in algebra (quadratics and review work) followed or preceded by solid geometry. It was our experience at our school that, when the third-year algebra came round, the students had forgotten most of their first-year work, so that it took all of six weeks to recover lost ground. So we began about five or six years ago to distribute algebra and geometry over the first two years, giving a considerable part of the time spent on geometry to work of a practical character, drawing a good deal, taking a number of propositions for granted, simply stating them and reasoning from them as if they were axioms. The results have been eminently satisfactory, to us at least, and I know that students taught on those lines were, at the beginning of the third year, superior to those taught in the old way. But when such a student, *at the end of his first year,* changes school and is told that he must go back and enter the first year in mathematics again because he has not covered enough ground to be admitted to the second year, it is not so pleasant. Few parents are qualified to understand pedagogical subtleties, even if they were inclined to listen to them, and what appears on the surface is warrant enough for

them to decide that their sons had been poorly taught: how else could they be behind others who began at the same point a year ago? However, improvement is impossible, unless there is breaking-away from old ways; and in the long run the fruits will fall to those who are at liberty to break away and who have the courage to do so.

If interest and love for the subject is of first importance, our next query will be how to arouse interest and awaken such love. First: At all stages beware of mechanical work and routine; strive to rouse thought. It must be *at all stages.* It must begin way down in the grades. When I test a new boy, I ask him: "How much is 5 times 67? If he begins: "5 times 7 is 35; carry 3; 5 times 6 is 30 and 3 is 33, makes 335," I know what training he has had. When I ask further: "How many times is ¾ contained in 15?" if he answers either not at all or, "Invert and multiply," I know all I want to know about him. With the real teacher it is head-work against finger-work all the time. We are still under the charm of the old fetish, *ciphering,* and many teachers that make an honest effort to give their students drill in mental work get no farther than teaching them to write their figures in the air insteal of on paper.

When the mechanics of written work are mastered, attention ought to be paid to short methods; even in mere computations brains ought to come into play all the time. In many computations, where we multiply numbers with several decimals, it is desirable to keep only a few in the result. It is stupid to figure the result out with all the decimals in the factors and then cut off the superfluous places after the determination of the decimal point. It is just as easy to determine the decimal point in the first place, from the integers, as the same law holds before the decimal point as behind. Then, after the decimal point is found, figure out only as many decimals as you want and drop the rest. I learned these things in the high school when I was fourteen years old, and I have practiced them ever since. There are other devices which there is not time to give, but which must be known to many.

It is one of the inherent difficulties in algebra to keep in

sight the connection between the letters and what they stand for. Evaluation of algebraic expressions, verification of equations, checking of results of addition and subtraction by substituting numerical values, help some. But with all that one can do, it is, I think, unavoidable for algebra, at least in the lower stages, to become largely a teaching of successive processes. And that is the reason why algebra has not nearly the culture value that it is possible to get out of geometry properly taught. The only place where you get out of mere processes into the realms of realities is where you reach problems leading to equations. I think it is a serious blemish on our algebra-teaching that so much importance is attached to processes and reductions and so little to the solution of problems. The colleges, at least many of them, foster this by demanding a degree of skill in the juggling of algebraic expression that exceeds the needs of the case. On the other hand, there is hardly one that dares put more than one problem on any examination paper, and I venture to assert that no student was ever refused the pass mark that omitted the problem, provided his juggling was even passably well done. In the schoolroom, however, this predominance of thimble-rig over thought is not specially felt as a deadening influence. Students at that age above all enjoy *doing;* they enjoy the learning of new processes, and the underlying thought-processes do not bother them much. It is the same stage of development that possesses a large verbal memory, that memorizes with ease, that is best suited to the study of languages, just because the acquisition of knowledge by imitation and memory is so much easier than by abstract thought, for which this age is not yet mature. Nevertheless, many times more time should be spent in problem-solving than is done now. We waste much time that could be made available for that. Neither I, nor anyone present, however far he may have gone in mathematics, has ever had occasion to find the H. C. F. or the L. C. M. by the process of division, except as an exercise in his schoolbook. We spend too much time in the reduction of complicated fractions and in the factoring of expressions containing more than four terms. It is interesting and instructive, it is said. I have no doubt that the processes of multiplication and division

with Roman numerals might be both interesting and instructive; why don't we do it? Many complicated expressions that our boys now labor over will be handled without difficulty later on, when greater maturity has increased their powers.

It is unfortunate that the conventional sequence of subjects almost precludes the taking-up of geometrical problems. I should not be afraid to anticipate some propositions, and treat them algebraically, both with numerical and literal values. A much wider field opens in the third year, after solid geometry has been studied. There is an endless variety of problems connecting algebra and geometry possible then; and I was pleased to find in a recent article by Mr. Lennes, of the Wendell Phillips High School, on secondary mathematics, which appeared in the May *School Review,* mention of the regular tetrahedron as a solid which I, like him, have found a fruitful source of interesting problems.

No opportunity should be lost to correlate the different mathematical subjects. I might say at this place that I see much more in the correlation of the mathematical subjects among themselves than in the wider scheme of correlating school mathematics with all possible phases of life. The only subject outside of geometry that seems to me at all promising is physics. But we study physics usually in the last or next to the last year of the high-school course, so that we can draw but few subjects of physics into the range of school mathematics without teaching them specifically, thus displacing the center of interest. Such subjects are: the relations between density, volume, and specific gravity, and the laws of parallel forces. I suppose you have all read the admirable article by Professor Myers in the October *School Review;* those who have not read it ought to. I must confess to some fear of confusing the mind of the student by the apparent complexity of the apparatus used; and if it were not for the greater ease with which Professor Myers' apparatus lends itself to lecture-table demonstration, I should prefer that which Professor Edwin H. Hall, of Harvard, has devised for the study of the same phenomena.

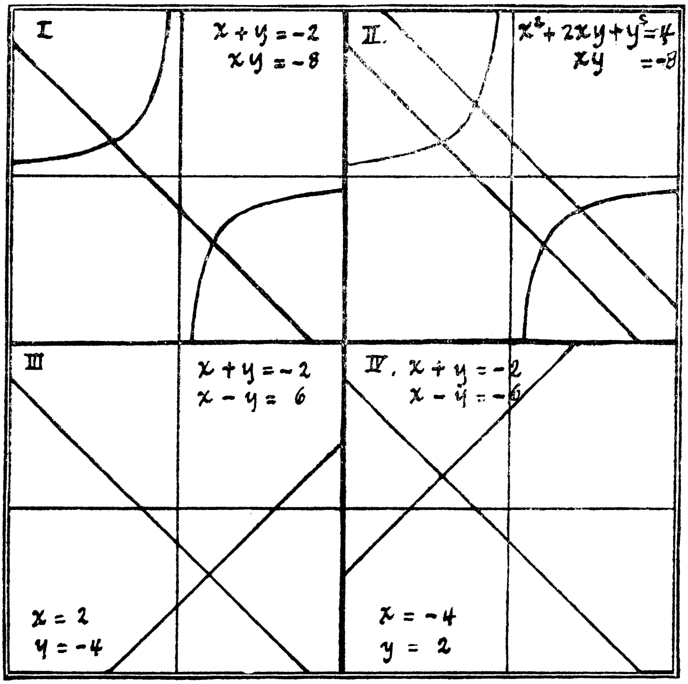
There is one point upon which I would lay great stress, namely, that the teaching of geometry should be liberated as

much as possible from the narrow limitations put upon it by purely logical deductive method. The fact that geometry is an art as well as a science, and was an art long before it became a science, has sometimes been lost sight of. We stand still too much under the influence of the powerful and absolutely unique impression made upon geometry by Euclid two thousand years ago. A student may go through any of our textbooks, and yet not be able to do anything with his geometry. It is only by solving innumerable problems that he learns to appreciate the meaning and the bearing of the propositions that he has studied and proved. I want especially to emphasize construction problems, which are too much neglected. They must really be solved, not merely by word of mouth and by reciting at the blackboard, but actually done on paper with ruler and compass. The construction should be nicely finished, in aesthetically attractive fashion. It is easy to arouse among the students a laudable ambition, a spirit of emulation as to who can produce the neatest work, which well repays whatever labor the teacher may have spent upon this aspect of their work. The metrical relations cannot be fixed in the memory beyond the peradventure of confusion without very many calculations, numerical and literal. The ignorance among most boys in regard to the metrical relations of the most interesting, perhaps, of all figures dealt with in plane geometry, the circle, is quite appalling, probably because the circle is treated last in the conventional scheme, at a time when everybody, teachers and students alike, are overloaded. We should not allow ourselves to be bound down by such artificial limitations. All the metrical relations of the circle can be made as clear as the most elaborate proof can make them by considerations that every student can follow, and after that there should be no difficulty in beginning work on the circle early enough to overcome once for all the confusion between πR^2 and $2\pi R$.

I want to say a few words on the graph. This is a comparatively new subject in this country, and it is quite plain that many teachers, and textbook-writers for that matter too, don't quite know what to make of it. Here as in other chapters of algebra the fruitfulness of the work depends altogether on what use you

make of it. Here as elsewhere it is possible to take it up in such a way that it is simply an addition to an already pretty full load, without any particular benefit.

In the first place, it is a better means than any yet devised for replacing the idea of *unknown quantity* by *variable quantity*. The former is comparatively barren; the latter is the mother of all further progress. It is the idea of functionality. It opens a new understanding of the solution of the equation, the root of an equation, the roots of simultaneous equations. It shows the student a way of getting at problems that are quite hard, and some of which are otherwise quite beyond his powers by any other method that he knows. Notwithstanding the apparent simplicity of the relation between time, rate, and distance, problems of motion are, as every teacher knows, hard for students to grasp. By means of the graph conditions can be simply stated and solutions arrived at in half the time and in a manner entirely delightful to the class. It is a very great pleasure to watch the tense attention, and at last the surprise and pleasure, on the faces of the class as the result appears. As a means for vivifying interest the graph is unsurpassed. The meaning of negative results in problems of motion can be made clearer than in any other way. For the solution of quadratic equations with one or two unknown quantities the graph is at its best. To me the conic sections have always been the source of considerable aesthetic pleasure. The solution of an equation is the solution of a riddle, and when it comes out, perhaps in the familiar form of a circle, or such a strange curve as the hyperbola, there is not a little pleasure in the result, quite apart from its algebraic meaning. After the idea has once been clearly grasped that the points whose coordinates fulfil the conditions imposed by the equation are limited to, or can move only on, such and such a line, a new meaning of the solution of the quadratic equation breaks out. By following out in detail the graphic figures corresponding to the successive operations performed in the solving of the equations, there appears an inkling of the reasons why certain things are done, why certain other things cannot be done, and why certain things, if done,



lead to too many or too few roots. A few diagrams that I have brought along with me will make my meaning clearer.

The first set of graphs illustrates the successive operations to be performed in the solution of the system of simultaneous equations

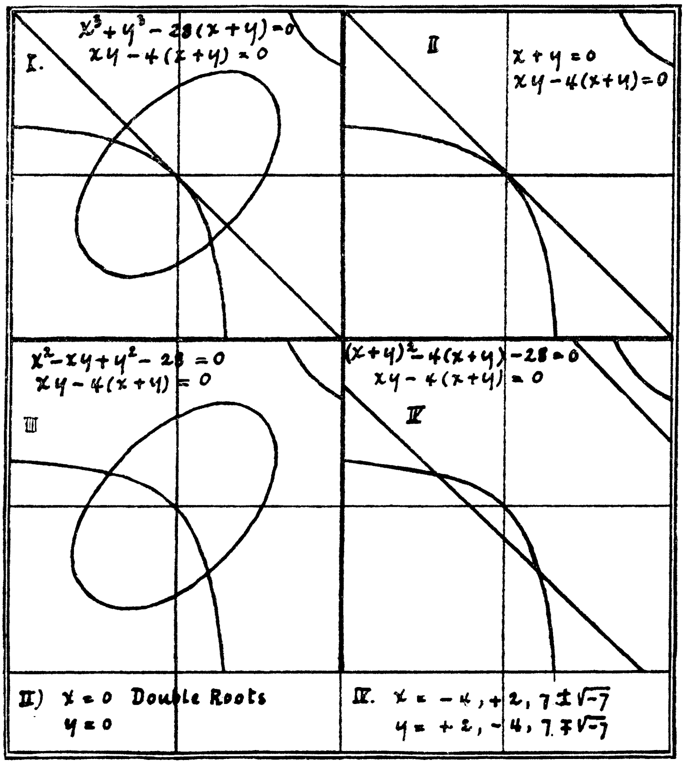
$$x+y=-2\,,$$
$$xy=\ 8\,;$$

the second illustrates in the same way the solution of the system

$$x^3+y^3-28\,(x+y)\ =0\,,$$
$$xy-4\,(x+y)\ =0\,.$$

The graphs of the latter are especially instructive. There ought to be six sets of solutions; two are real and different (intersection), two real and equal (tangency), two imaginary (failure to intersect).

All this is pure mathematics. But the graph allows us also to solve applied problems that could not easily be reached by algebra



at this stage. I anticipate a little physics and tell the class that a body falls during the first second 16 feet, during the next 4 times 16, and so on. Also that 32 feet is in physics denoted by the letter g. It does not take long to express the law stated numerically in literal notation:

$$s=\tfrac{1}{2}\,gt^2$$

A body is thrown into the air at a certain angle with the horizontal; at what point will it strike the ground? What is the greatest height it reaches? Plot its path. Next trace the course of another projectile, making the same angle with the verticle as the first does with the horizontal. It is interesting as well as surprising to the class that the two reach the ground at the same distance. It is at the same time a bit of information on ballistics that is not likely to be forgotten. Another problem: Of all the

rectangles with a given perimenter $2p$ which has the greatest area? The formula for the area is

$$y=x\ (p-x)\ ;$$

or

$$y=px-x^2,$$

the graph of which is, of course, a parabola with vertical axis, convexity turned upward. It gives the information desired for a particular case. A simple algebraic transformation gives to the equation the form

$$y=p/4-(p/2-x)^2$$

from which it appears that the maximum occurs when

$$x=\frac{p}{2}\ ;$$

that is, when the rectangle is a square. Armed in this way, both with algebra and the graph, the student may easily investigate other problems on maxima and minima, which do not lead to equations higher than the second degree, such as the converse of the one just mentioned: The area being given, what must be the relation between base and altitude, so that the rectangle may have the smallest perimeter?

But the graph is not limited to equations of the second degree. Curves of higher order are a trifle more laborious, but not essentially harder. Many problems from plane and solid geometry, even in cases where the equation to which they lead is of higher order than can be managed at this stage, can readily be treated by means of the graph, at least for definite numerical values.

Before closing I cannot forbear saying a few words regarding propositions for improvement in the teaching of mathematics that have been made abroad, especially by Professor Perry and by Professor Felix Klein. While Mr. Perry pleads for improvement and reform for the benefit of *all boys,* it becomes quite plain to anyone who has read the series of articles that he has published under the title *England's Neglect of Science,* his essay read before the British Association, and his *Syllabus of Mathematics,* that he pleads in the first, second, and third place for students such

as he has in his own institution—boys of the age of our high-school boys, following purely technical studies of much more ambitious scope than our manual-training schools pursue. Many of his proposals are extremely radical, and he presents them in a brilliant and most incisive way. The discussion of the subject of improvement in the teaching of mathematics held at the meeting of the British Association at Glasgow in 1901 is most interesting. It is published by The Macmillian Company, and costs but 65 cents. Mr. Perry's *Syllabus of Practical Mathematics* is published by Wyman & Sons (London, 1901). The pamphlet entitled *England's Neglect of Science,* though less interesting than the other two mentioned, is well worth reading. It is published by Fisher Unwin (London, 1900). All three are inexpensive. I may mention the fact that these, and a good many most interesting works on mathematics and the pedagogy of mathematics, are to be found in the Crerar Library, an institution which is collecting for us rich materials for investigation in science and mathematics, that no one can afford to overlook who can command the time for studying at a reference library. Professor Klein's essays, and those of his collaborators in his field, are well worth reading also. They are much more scholarly than Mr. Perry's, but their propositions concern more especially the higher classes of the German *Gymnasium* and those of the *Realschule* than schools of the grade we mean when we speak of secondary schools.

Professor Perry's books are instructive also from another standpoint, because of the light they throw incidentally on conditions in England. The reader is especially surprised at the extraordinary state of backwardness in the teaching of geometry that seems to prevail in England, owing to the, to us inconceivable, conservatism in the retention of Euclid as the sole and only textbook in geometry. Hence an everywhere prevailing impression that geometry is a subject supremely difficult, that can be mastered only by a few select minds. This crops out in the course of the discussion in the most amusing way. Hence also in the propositions for reform a curious underestimation of the educational value of geometry and what seems to us an overvaluation,

at least at this stage, of algebraic analysis. A committee of the British Association, appointed to report on the improvement in the teaching of mathematics, says in one place: "It may be felt convenient to retain Euclid; but perhaps *the amount to be memorized* might be curtailed." All through the discussion it appears that to all, or almost all, speakers it seems axiomatic that the same textbook must be used by all the schools in England, if geometry is to be taught at all; how else could there be examinations in geometry? The president of the section for mathematics and physics says: "I have tried to find out from those prominent in the Society for the Improvment of Mathematical Teaching what it really proposes as a substitute for Euclid, but without success." Professor Sylvanus Thompson says: "The teaching of arithmetic should follow a certain course, and pupils ought to be taught how to differentiate and to integrate simple algebraic expressions, before we attempt to teach them geometry and these other complicated things." Professor Perry, speaking of elementary geometry-teaching, says: "A good teacher will *occasionally* introduce demonstrative proof as well as mere measurement."

Surely, the English reformers' needs are not our needs; and while we may read their books with profit, and follow their struggles with sympathy, we must fight our own battles, and shape our ends according to our own needs.